THE DISTRIBUTION OF TURBULENT ENERGY IN FREE JETS

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We discuss the probable form of the distribution of energy for pulsating motion in self-similar free turbulent jets of an incompressible fluid, based on the considerations of similarity. The form of the integral invariants for self-similar flow is indicated.

§1. There is considerable interest in investigating the distribution of the pulsation energy $E = 1/2 \sum_{i=1}^{3} u_i'^2$ as well as the balance equations for this energy, in connection with the study of turbulent motion [1, 2]. A number of papers [3-5], etc.) has recently been published, and in these attempts have been made – within the framework of semiempirical theories of turbulence – to integrate the balance equations for the pulsating energy in the boundary layer. The simplifications and hypotheses introduced in the solution cannot always be regarded as self-evident; however, there is no doubt that they make it possible to achieve satisfactory agreement with experiments in a constructive manner when empirical information (compared with the information required for the calculation of the average flow) is taken from the experiment.

For the theory of turbulent jets (as for a turbulent boundary layer as a whole) development along such lines promises, basically, the ability to establish the relationship between the characteristics of averaged and pulsating flows. At the same time, it may be possible to use the data on the mechanism of the process that are available from the theory of turbulent jets to validate the methods of semiempirical calculations.

The foregoing statement should not be regarded as excessively illusory. We can draw an extremely important and promising conclusion from the work done in recent years in the Soviet Union and abroad in connection with investigations into the pulsation structure of turbulent jets of an incompressible fluid. We are dealing here with the fact that direct thermoanemometric measurements of the average pulsation product $\overline{u'v'}$, within the limits of experimental error, give results that coincide with the value of τ_f/ρ – the tangential frictional stress determined by calculation from the equation of the free boundary layer:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y} \left(y^k \frac{\tau_f}{\rho} \right), \tag{1}$$

$$\frac{\partial u y^k}{\partial x} + \frac{\partial v y^k}{\partial y} = 0 \tag{2}$$

($k = 0$ for 1, respectively, for a plane or axisymmetric jet), if into the first of these equations, solved for τ_f, we substitute the results from the measurements of the average velocity, determined experimentally.

This coincidence testifies, first of all, to the suitability of the approximate equations (1) and (2) for purposes of describing the averaged motion and, secondly, as to the propriety of utilizing the calculation methods and schemes from jet theory [6], which are virtually exact in showing the distribution of the average quantities (consequently, of turbulent flow as well).

Direct measurements (in [10]) have also confirmed experimentally that the calculation of the transverse component of the averaged velocity v from (2), i.e., within the framework of boundary-layer theory, leads to virtual agreement with experiment.

To find the distribution of E – the average pulsation energy – in addition to (1) and (2), we should integrate the energy balance equation

$$u \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} = \frac{1}{y^k} \frac{\partial}{\partial y} \left[y^k \left(\overline{v'E'} + \overline{v'p'} \right) \right] - \overline{u'v'} \frac{\partial u}{\partial y} + \varepsilon \tag{3}$$

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(where ε is the dissipative term). To integrate (3) we have to specify the expressions for the diffusion of pulsation energy, based on certain considerations (usually on the basis of dimensionality, as per Kolmogorov), and we also have to specify the dissipation of the first and third terms in the right-hand member of Eq. (3). It is only the generation of energy (the second term on the right) that is entirely specified by the solution of (1) and (2).

As a result of integrating (3) under the corresponding boundary conditions (the solution of (1) and (2) is assumed to be known) we are able to find the energy distribution $E = E(x, y)$ and its convection.

Naturally, the first attempts at direct solution of the problem were directed at the simplest self-similar jets, i.e., those motions initiated by turbulent jet sources at a considerable distance from the outlet of the jet. An interesting attempt along these lines was undertaken in [5] in connection with a self-similar plane jet of an incompressible fluid. It is based on notation of the diffusion of pulsation energy in the form of a single gradient term and a modified expression for dissipation, which contains (as does the expression for diffusion) the turbulent viscosity ν_t from the momentum equation (1) as the transfer coefficient. As a result, on introduction of the second empirical constant [5] it became possible to bring the results from the numerical integration of (3) into agreement with the experimental data from [9].

Below we describe a somewhat different attempt – a more general approach to the same self-similar problem – which might be useful in other cases.

§2. For the self-similar region of flow, as is well known, there exists a universal profile for the average velocity in turbulent jets:

$$u = u_m F(\varphi), \quad u_m = Ax^{\alpha} \quad \varphi = Byx^{\beta}, \tag{4}$$

where $B \equiv 1/a$ is an experimental constant; A is a constant determined from the initial condition (based on the specified momentum flux); $\beta = -1$ for plane ($\alpha = -1/2$) and axisymmetric ($\alpha = -1$) turbulent jets; u_m is the velocity at the axis; finally, $F(\varphi) = \mathrm{ch}^{-2}\varphi$ for a plane and $F(\varphi) = (1 + \varphi^2/8)^{-2}$ for an axisymmetric jet, when the solution is based on the so-called new Prandtl formula ($\nu_t = bu_m$, see for example [6]). Since $u_m \sim x^{\alpha}$, it follows from (1), after it has been reduced to self-similar form, that $\overline{u'v'} \sim x^{2\alpha}$, i.e., $\overline{u'v'} \sim 1/x$ for a plane jet and $\sim 1/x^2$ for an axisymmetric jet.

As follows from the investigation of the pulsation structure of turbulent jets, at a sufficiently great distance from the sources (apparently greater than is required for self-similarity on the basis of average velocity) we note the establishment – correct for practical purposes – of the universal profile for the average pulsation characteristics $\sqrt{\overline{u'^2}}$, $\overline{u'v'}$, E, etc. Consequently, it may be assumed for a self-similar jet that

$$E = E_m(x) f(\varphi), \quad E_m \sim x^s \tag{5}$$

and Eq. (3) can be rewritten in self-similar form, in which case it will be independent of the coordinates x and y separately (while dependent only on the reduced coordinate $\varphi = By/x$). However, if we know the above-indicated relationship $\overline{u'v'} \sim x^{2\alpha}$, and also that the derivatives $\partial u/\partial x \sim x^{\alpha-1}$, it is possible without detailed notation of the equation to establish the relationship between the quantity between E in the self-similar region of flow and the x coordinate. From these considerations we see that $E_m \sim x^s \sim x^{2\alpha}$, i.e., $s = -1$ for a plane jet and $s = -2$ for a circular jet. Knowing the value of the self-similarity constant s for the pulsation energy, from the same considerations of dimensionality we can establish the form of the integral invariant for (3). Indeed, if we write the expression

$$\int_0^\infty uE^z y^k dy = u_m E_m^z x^{1+k} \int_0^\infty F(\varphi) f^z(\varphi), \tag{6}$$

where z is the exponent – as yet unknown – it follows from the universality of the profiles $F(\varphi)$ and $f(\varphi)$ that the integral on the right-hand side represents some number.

It is obvious that for the preintegral factor $u_m E_m^z x^{1+k} \sim x^{\alpha+zs+1+k}$ also to be independent of the coordinate, the value of the exponents z should be set equal to

$$z = -\frac{\alpha+1+k}{s} = -\frac{\alpha+1+k}{2\alpha} = \frac{1}{2} \tag{7}$$

for plane and axisymmetric jets. In this case, the following equation is valid:

$$\frac{d}{dx}\int_0^\infty u\sqrt{E}\, y^k dy = 0, \quad \int_0^\infty u\sqrt{E}\, y^k dy = \text{const.} \tag{8}$$

The quantity proportional to $E^{1/2}$ is denoted V_E, and essentially it characterizes the mean quadratic pulsation velocity

$$V_E = \sqrt{\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{3}} = \sqrt{\frac{2}{3}E}$$

(in the case of local isotropy for the pulsation of velocity $V_E = \sqrt{\overline{u'^2}} = \sqrt{2E}$).

Thus, along a self-similar jet (plane or axisymmetric) the flux of magnitude V_E will remain constant:

$$\int_0^\infty uV_E y^k dy = \text{const.} \tag{9}$$

Having determined the integral invariant of the problem, for the pulsation velocity V_E we can write the transfer equation in the form

$$u\frac{\partial V_E}{\partial x} + v\frac{\partial V_E}{\partial y} = \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k D_E \frac{\partial V_E}{\partial y}\right), \tag{10}$$

where D_E is the turbulent diffusion coefficient for the quantity V_E. The validity of this equation (given reasonable hypotheses with respect to the coefficient D_E) can be demonstrated by proceeding from the reciprocal. For this purpose, let us write the boundary conditions for the turbulent energy E (or V_E), and at the same time for the velocity

$$\begin{aligned} v = 0,\ \frac{\partial u}{\partial y} = \frac{\partial E}{\partial y} = 0 \quad \left(\frac{\partial V_E}{\partial y} = 0\right) \text{ when } y = 0; \\ u = \frac{\partial u}{\partial y} = E = \frac{\partial E}{\partial y} = 0 \quad \left(V_E = \frac{\partial V_E}{\partial y} = 0\right) \text{ when } y = \infty. \end{aligned} \tag{11}$$

With consideration of these boundary conditions, if we integrate (10) in a form derived by means of the continuity equation (2)

$$\frac{\partial uV_E y^k}{\partial x} + \frac{\partial vV_E y^k}{\partial y} = \frac{\partial}{\partial y}\left(y^k D_E \frac{\partial V_E}{\partial y}\right), \tag{10a}$$

across the entire boundary layer (from y = 0 to y = ∞), after simplification we obtain the same invariant: $\int_0^\infty uV_E y^k dy = \text{const}$.

However, from a comparison of (1) and (10), if in the first of these we express $\tau_f/\rho = \nu_t\, \partial u/\partial y$, where ν_t is the turbulent viscosity, and if we also denote the ratio $\nu_t/D_E = \varkappa$ (a kind of turbulent Prandtl number or the transfer V_E), we see that the profiles for V_E and u are associated with each other by a simple relationship of the form

$$\frac{V_E}{V_{Em}} = \left(\frac{u}{u_m}\right)^\varkappa \tag{12}$$

in analogy with the profiles of temperature and velocity in nonisothermal jets. As regards the value of the coefficient $\varkappa$, in first approximation assumed here to be constant, it becomes possible on the basis of the known experimental data [12] to assume that $\varkappa \approx 1$, or hardly different from unity. However, the value of $\varkappa$ is in need of further refinement.

Thus, retaining, for example, the same scheme of solution for the asymptotic layer and the new Prandtl formula ($\nu_t = bu_m$), we find the distribution of the pulsation energy in the form

for a plane jet

$$E = \text{const}\, x^{-1} \operatorname{ch}^{-2\varkappa} \varphi, \tag{13}$$

for an axisymmetric jet

$$E = \text{const}\, \bar{x}^2 \left(1 + \frac{\varphi^2}{8}\right)^{-2\varkappa}. \tag{14}$$

The value of the constants in these formulas is associated with the invariants $\int_0^{-\infty} u^2 y^k dy$ and $\int_0^\infty uV_E y^k dy$ of the problem.

Having determined the distribution of the pulsation energy, we can also calculate its convection in accordance with the expression $u\,\partial E/\partial x + v\,\partial E/\partial y$. Thus, Eq. (3) will retain two undefined terms – the diffusion and dissipation of energy. For example, assuming an expression for diffusion in the form

$$\frac{1}{y^k}\frac{\partial}{\partial y}\left[y^k\left(\overline{v'E'}+\overline{v'p'}\right)\right]=\frac{\sigma_E}{y^k}\frac{\partial}{\partial y}\left(y^k\nu_{\text{T}}\frac{\partial E}{\partial y}\right), \tag{15}$$

where σ_E is yet another "turbulent Prandtl number," generally different from unity, we can find the dissipation from the difference. Of course, other means are possible to determine the balance (in particular, the use of the relationship between the right-hand members of (3) and (10), etc.).

For the self-similar flow region (average and pulsation flow) the profiles of the individual components of the turbulent pulsations in velocity, i.e., $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, $\sqrt{\overline{w'^2}}$, $\overline{u'v'}$, etc., must also be universal. Therefore, for each of these components we can write analogous particular invariants of the form

$$\int_0^\infty u\sqrt{\overline{u'^2}}\,y^k dy=\text{const},$$

as well as other invariants (see below) and expressions of the form

$$\frac{\sqrt{\overline{u_i'^2}}}{\sqrt{\overline{u_{mi}'^2}}}=\left(\frac{u}{u_m}\right)^{\varkappa}.$$

§3. Let us make several remarks with respect to these results. First of all, we note that the above-derived invariant for the self-similar flow $\int_0^\infty uV_E y^k dy=\text{const}$ is not unique. The same consideration (dealing with the independence of the x coordinate) lead to the general form of the invariant $\int_0^\infty u^{z_1}V_E^{z_2}y^k dy=\text{const}$, where z_1 is an arbitrary number, and $z_2=2-z_1$. As examples we can cite the integrals $\int_0^\infty u^2y^k dy$, $\int_0^\infty Ey^k dy \sim \int_0^\infty V_E^2 y^k dy$, $\int_0^\infty u^3V^{-1}y^k dy$, and numerous others. Each of these integrals will retain its value along the self-similar segment of the jet.

We can point to other families of integral invariants, for example, of the form $\int_0^\infty u^{z_3}\left(\int_0^y V_E^{z_4}y^k dy\right)y^k dy=\text{const}$ (for instance, $\int_0^\infty u^2\left(\int_0^y Ey^k dy\right)y^k dy=\text{const}$ or $\int_0^\infty E\left(\int_0^y u^2y^k dy\right)y^k dy=\text{const}$ for $z_3=z_4=2$, etc.), where z_3 is an arbitrary number, and $z_4=4-z_3$, etc.

This result (and those similar to it) is a consequence of the transformation of self-similarity and pertains to any quantity whose distribution is subject to a formula of the form of (6).

The existence of these invariants makes it possible with comparative ease to use the experimental data to test the validity of the fundamental hypotheses and primarily the self-similarity of the average and pulsation motion. However, detailed and reliable data containing measurements of all three velocity pulsations, etc., are presently not available. For an axisymmetric jet the relationship $E_m \sim x^{-2}$, as well as the approximate similarity of the V_E and u profiles is apparently confirmed experimentally [10-12]. The form of the individual terms in the energy-balance equation derived by calculation [11] agrees qualitatively with experiments.

For a plane jet, on the basis of the data from [9] and from calculation [5], in the $\sqrt{\overline{u'^2}}$ pulsation velocity profile on the segment that is close to the self-similar, the minimum is retained on the jet axis (in the case of a round jet it is absent [7]). It is possible that in these tests the self-similar profile of the pulsation energy* has not yet been completely established because of the presence of a weak cocurrent, or for some other reason. A reliable conclusion in this connection can be drawn only after special tests are carried out and if detailed measurements are made of each of the three components of the pulsation velocity, etc.

*This is supported by results from the unique Mikhasenko experiment involving a plane jet under the action of a mechanical turbulizer [13] – with an elevated initial turbulence level the minimum for $\sqrt{\overline{u'^2}}$ virtually disappears on the axis of the plane jet. (In tests with an axisymmetric jet – with and without a turbulizer – the constancy of the integral $\int u\sqrt{E}dy$ has been confirmed.)

Let us also point out that Eq. (3), at first glance, admits of yet another variant for the self-similar solution. We are speaking of an ideal turbulent jet for which two pairs of terms in the balance equation are separately offset at each point: the generation and dissipation of energy, on the one hand, and convection and diffusion on the other hand. In this case, the local power of the sources and sinks at any point on the jet must be equal to zero. In this event only two terms – convections and diffusions – are retained in (3) and it becomes similar to (1). However, in this case we must have $E_m \sim x^{\alpha}$ on the self-similar segment of the flow (rather than $x^{2\alpha}$, as derived above) and we would have the invariant $\int_0^\infty uEy^k dy = \mathrm{const}$ (instead of $\int_0^\infty u\sqrt{E}y^k dy = \mathrm{const}$). This does not agree with the self-similarity of (1) for the average velocity and, apparently, contradicts the well-known jet experiment. The further development and accumulation of experimental data will demonstrate whether or not the concept of an ideal pulsation energy for a turbulent jet (or of some other ideal turbulent flow) with locally compensated sources and sinks – which is useful for approximate evaluation – exhibit only theoretical significance or whether a motion that is self-similar with respect to the pulsation-energy profile under certain conditions can exist as an asymptotic motion.

Let us dwell briefly on the nonself-similar problem. In the general case we should be speaking of calculating the variations in the pulsation-energy profile along the entire jet, beginning from the specified initial profile to the developed self-similarity. It is precisely this formulation of the problem that best corresponds to the experiment (see, for example, [10, 11], etc.).

In such a general form the solution of the problem becomes extremely complex. With a monotonic variation in the average velocity (u_m diminishes in the direction of the jet) the pulsation energy E_m at the axis initially increases, subsequently passing through a maximum, and then diminishing. Thus, at the beginning of the jet the quantities u_m and E_m vary oppositely with increasing distance from the source, while at the end the variation is identical. It is therefore difficult to collect uniform expressions for the components in the pulsation-energy balance equation that are suitable for the entire jet. As regards the use of the method for the equivalent problem from the theory of heat conduction [6, 14], which has demonstrated its applicability in the calculation of average characteristics for nonself-similar flows, it may prove to be useful for the basic segment of the jet, i.e., the flows beyond the maximum E_m.

A more complete conclusion can be drawn only after thorough and systematic tests are performed.

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